

On extension of partial orders to total preorders with prescribed symmetric part¹

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Abstract For a partial order \preceq on a set X and an equivalency relation S defined on the same set X we derive a necessary and sufficient condition for the existence of such a total preorder on X whose asymmetric part contains the asymmetric part of the given partial order \preceq and whose symmetric part coincides with the given equivalence relation S . This result generalizes the classical Szpilrajn theorem on extension of a partial order to a perfect (linear) order.

Key words partial order, preorder, extension, Szpilrajn theorem, Dushnik-Miller theorem.

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Let X be an arbitrary nonempty set and let $G \subset X \times X$ be a binary relation on X . A binary relation $G \subset X \times X$ is called a *preorder* if it is reflexive ($(x, x) \in G \forall x \in X$) and transitive ($(x, y) \in G, (y, z) \in G \Rightarrow (x, z) \in G \forall x, y, z \in X$). If in addition a preorder $G \subset X \times X$ is antisymmetric ($(x, y) \in G, (y, x) \in G \Rightarrow x = y \forall x, y \in X$), then it is called a *partial order*. A total partial order is called a *perfect* (or *linear*) *order*. (A binary relation $G \subset X \times X$ is *total* if for any $x, y \in X$ either $(x, y) \in G$ or $(y, x) \in G$ holds.) In the sequel, a preorder will be preferably denoted by the symbol \preceq whereas a partial order as well as a perfect order by the symbol \preceq .

One of the key results of the theory of ordered sets is the following theorem proved by E. Szpilrajn in 1930 [1].

Theorem 1 (E. Szpilrajn [1]) *For every partial order $\preceq \subset X \times X$ there exists a perfect extension, i. e., there exists a perfect order $\preceq' \subset X \times X$ such that $\preceq \subset \preceq'$. Moreover, for any pair of elements $a, b \in X$ such that $(a, b) \notin \preceq$ and $(b, a) \notin \preceq$ a perfect extension $\preceq' \subset X \times X$ for the partial order \preceq can be chosen in such a way that $(a, b) \in \preceq'$.*

In 1941 E. Dushnik and B. Miller proved the following strengthening of the Szpilrajn theorem.

Theorem 2 (E. Dushnik, B. Miller [2]) *Every partial order $\preceq \subset X \times X$ is the intersection of all its perfect extensions.*

In the recent literature the Szpilrajn theorem and the Dushnik–Miller theorem and their proofs can be found in the monographs [3,4]. The generalizations of the Szpilrajn theorem to the case when partial orders and perfect orders extending them are defined on groups, rings and some other algebraic systems and are compatible with their algebraic operations are presented in the monograph of L. Fuchs [5]. Due to the duality between compatible perfect orders defined on a real vector space X and semispaces of X (the cones of positive elements of compatible perfect orders are complements of semispaces at zero) it follows from the results of V. Klee devoted to semispaces [6] that any compatible partial order defined on a real vector space X can be extended to a compatible perfect

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order. For relations defined on topological spaces the conditions under which there exist continuous total preorders extending partial orders were obtained by G. Bosi and G. Herden [7, 8]. The results of the studies devoted to the existence of utility functions for partial orders (see [3, 9] as well the monographs [10, 11] and bibliography cited there) can also be considered as generalizations of the Szpilrajn theorem.

Every binary relation G on X can be presented as the disjoint union $G = P_G \cup S_G$ ($P_G \cap S_G = \emptyset$) of its asymmetric part $P_G := \{(x, y) \in G \mid (y, x) \notin G\}$ and its symmetric part $S_G := \{(x, y) \in G \mid (y, x) \in G\}$. If G is a preorder then its symmetric part S_G is reflexive, symmetric and transitive and, consequently, in that case S_G is an equivalency relation on X , which is reduced to the equality relation when G is a partial order. The asymmetric part of a preorder (and, in particular, the asymmetric part of a partial order) is an asymmetric and transitive binary relation. On the other hand, the union of any asymmetric and transitive binary relation with the equality relation is a partial order. Thus, there exists the one-to-one correspondence between partial orders and asymmetric and transitive binary relations. Note that different preorders can have the same asymmetric part.

Let \preceq be a partial order on X and S an equivalency relation defined on the same set X .

A total preorder $\preceq \subset X \times X$ will be referred to as a *total preorder S -extension of the partial order \preceq* if the asymmetric part of \preceq contains the asymmetric part of the given partial order \preceq and the symmetric part of \preceq coincides with the equivalency relation S , that is, if $P_{\preceq} \subset P_{\preceq}$ and $S_{\preceq} = S$.

The main purpose of this paper is to derive for a given partial order \preceq and a given equivalency relation S a necessary and sufficient condition for the existence of a total preorder S -extension of \preceq .

In the case when S is the equality relation on X , i. e., when $S = E := \{(x, y) \in X \times X \mid x = y\}$, due to the Szpilrajn theorem, such an extension exists for any partial order \preceq . As it will be shown below in the general case the required extension exists if and only if the partial order \preceq and the equivalency relation S are compatible in some way. Thus the main results of the paper can be considered as a generalization of the Szpilrajn theorem.

Let us begin with consideration of a particular case. Assume that a partial order \preceq and an equivalency relation S hold the additional condition

$$P_{\preceq} \circ S = S \circ P_{\preceq} = P_{\preceq} \quad (1)$$

(the symbol \circ denotes the composition of binary relations).

It immediately follows from (1) that the union $\preceq \cup S$ is a preorder on X the symmetric part of which coincides with S . Let X/S be the quotient of X with respect to the equivalency relation S and let T be the quotient of the preorder $\preceq \cup S$ with respect to S . Since T is a partial order on X/S , due to the Szpilrajn theorem, T can be extended to a perfect order Q on X/S . Setting $x \preceq y \iff [x]_S Q [y]_S$ (here $[x]_S$ and $[y]_S$ stands for the equivalency classes of S containing x and y , respectively) we obtain the total preorder \preceq on X which is a total preorder S -extension of \preceq .

Thus the following theorem generalizing both the Szpilrajn theorem and the Dushnik–Miller theorem is true.

Theorem 3 *Let \preceq be a partial order on X . For any equivalency relation S on X which satisfies condition (1), there exists a total preorder S -extension of the partial order \preceq .*

Moreover, for any pair of points $a, b \in X$ such that $(a, b) \notin \preceq \cup S$ and $(b, a) \notin \preceq \cup S$, there exists a total preorder \preceq which is a total preorder S -extension of the partial order \preceq and $(a, b) \in \preceq$.

The intersection of all total preorder S -extensions of a preorder \preceq coincides with the preorder $\preceq \cup S$.

Along with each relation of preorder \preceq we will consider the *indifference relation* $I_{\preceq} := \{(x, y) \in X \times X \mid (x, y) \notin P_{\preceq}, (y, x) \notin P_{\preceq}\}$ corresponding to \preceq . In the general case the indifference relation

I_{\sim} is reflexive and symmetric, i. e. I_{\sim} is a tolerance relation. An indifference relation I_{\sim} is in addition transitive (and, consequently, it is an equivalency relation in this case) if and only if P_{\sim} is negatively transitive (it means that the negation of P_{\sim} , i. e. the relation $(X \times X) \setminus P_{\sim}$, is transitive). It immediately follows from the definition of I_{\sim} that every pair of points $x, y \in X$ satisfies one and only one of the following three alternatives: $(x, y) \in P_{\sim}$, $(y, x) \in P_{\sim}$ and $(x, y) \in I_{\sim}$.

Another binary relation on X generated by a preorder \sim is the equipotency relation R_{\sim} , which is defined by

$$(x, y) \in R_{\sim} \iff \{z \in X \mid (x, z) \in I_{\sim}\} = \{z \in X \mid (y, z) \in I_{\sim}\}.$$

It is not hard to verify that R_{\sim} is an equivalency relation on X with $R_{\sim} \subset I_{\sim}$. The equipotency relation R_{\sim} is equal to the indifference relation I_{\sim} , i. e. $R_{\sim} = I_{\sim}$, if and only if P_{\sim} is negatively transitive (or, equivalently, if and only if I_{\sim} is transitive).

Proposition 1 *Let \sim be a preorder on X . An equivalency relation $S \subset X \times X$ holds the equalities $P_{\sim} \circ S = S \circ P_{\sim} = P_{\sim}$ if and only if $S \subset R_{\sim}$.*

Proof Assume that an equivalency relation S satisfies the equalities $P_{\sim} \circ S = S \circ P_{\sim} = P_{\sim}$ and let $(x, y) \in S$. The alternative $(x, y) \in P_{\sim}$ is impossible, because otherwise it would follow from $(y, x) \in S$ and from the equality $P_{\sim} \circ S = P_{\sim}$ that $(x, x) \in P_{\sim}$, but it contradicts the asymmetric property of P_{\sim} . Similarly we can show that the alternative $(y, x) \in P_{\sim}$ is also impossible. Hence, $(x, y) \in I_{\sim}$.

Let us prove that in fact $(x, y) \in R_{\sim}$. Choose an arbitrary element $z \in X$ such that $(x, z) \in I_{\sim}$ and consider the ordered pair $(y, z) \in X \times X$. The alternatives $(y, z) \in P_{\sim}$ and $(z, y) \in P_{\sim}$ are impossible, because otherwise it would follow from $(x, y) \in S$ and $P_{\sim} \circ S = S \circ P_{\sim} = P_{\sim}$ that $(x, z) \in P_{\sim}$, which contradicts the choice of z . Hence, $(y, z) \in I_{\sim}$ and, consequently, $\{z \in X \mid (x, z) \in I_{\sim}\} \subset \{z \in X \mid (y, z) \in I_{\sim}\}$. The converse inclusion is proved in the similar way. Thus, $\{z \in X \mid (x, z) \in I_{\sim}\} = \{z \in X \mid (y, z) \in I_{\sim}\}$ and we conclude from the definition of R_{\sim} that $(x, y) \in R_{\sim}$.

To prove the converse statement we note that the inclusions $P_{\sim} \subset P_{\sim} \circ S$ and $P_{\sim} \subset S \circ P_{\sim}$ follow from the reflexivity of the relation S . So we need to prove the converse inclusions. Let $(x, y) \in S \circ P_{\sim}$. Then there exists an element $z \in X$ such that $(x, z) \in S$ and $(z, y) \in P_{\sim}$. Assume that $(y, x) \in P_{\sim}$. Due to the transitivity of P_{\sim} we conclude from $(z, y) \in P_{\sim}$ that $(z, x) \in P_{\sim}$, which contradicts $(x, z) \in S \subset R_{\sim} \subset I_{\sim}$. Consequently, $(y, x) \notin P_{\sim}$. The assumption $(x, y) \in I_{\sim}$ also leads to a contradiction. Indeed, for $(x, z) \in S \subset R_{\sim}$ we have due to the definition of R_{\sim} that $(x, y) \in I_{\sim}$ implies $(z, y) \in I_{\sim}$, which contradicts $(z, y) \in P_{\sim}$. Hence, $(x, y) \notin I_{\sim}$ and, consequently, the alternative $(x, y) \in P_{\sim}$ is uniquely possible. Thus, $S \circ P_{\sim} \subset P_{\sim}$.

The inclusion $P_{\sim} \circ S \subset P_{\sim}$ is proved in the similar way. \square

Corollary *For every partial order \preceq on the set X and every equivalency relation S on the same set X such that $S \subset R_{\preceq}$ there exists a total preorder S -extension of \preceq .*

Let us consider now the general case, that is the case when a partial order \preceq and an equivalency relation S do not necessarily satisfy equalities (1). We begin with the following (evident) necessary condition for the existence of a total preorder S -extension of a partial order \preceq .

Theorem 4 *Let S be an equivalency relation on a set X . If for a partial order $\preceq \subset X \times X$ there exists a total preorder S -extension then $S \subset I_{\preceq}$.*

Proof Let $(x, y) \in S$. If $(x, y) \in P_{\preceq}$ or $(y, x) \in P_{\preceq}$, then for any total preorder S -extension \sim of a partial order \preceq we would have $(x, y) \in P_{\sim}$ or $(y, x) \in P_{\sim}$, respectively. However, since $P_{\sim} \cap S = \emptyset$, the both alternatives are impossible and, hence, $(x, y) \in I_{\sim}$. \square

Proposition 2 *Let S be an equivalency relation on a set X and \preceq a partial order defined on*

the same set X . Then $S \subset I_{\preceq}$ if and only if the composition $S \circ P_{\preceq} \circ S$ is irreflexive.

Proof Recall that the irreflexivity of the binary relation $S \circ P_{\preceq} \circ S$ means that $(x, x) \notin S \circ P_{\preceq} \circ S$ for all $x \in X$.

Let $S \subset I_{\preceq}$. Assume that contrary to the assertion of the proposition the composition $S \circ P_{\preceq} \circ S$ is not irreflexive. The latter means that $(x, x) \in S \circ P_{\preceq} \circ S$ for some $x \in X$. Due to the definition of the composition we can find such elements $y, z \in X$ that $(x, y) \in S$, $(y, z) \in P_{\preceq}$ and $(z, x) \in S$. Since S is transitive, it follows from $(z, x) \in S$ and $(x, y) \in S$ that $(z, y) \in S$. Hence, since S is symmetric, $(y, z) \in S \cap P_{\preceq}$, which contradicts $I_{\preceq} \cap P_{\preceq} = \emptyset$. This proves that $S \subset I_{\preceq}$.

Assume now that the composition $S \circ P_{\preceq} \circ S$ is irreflexive, but the inclusion $S \subset I_{\preceq}$ is not the case. Then there exists $(x, y) \in S$ such that $(x, y) \notin I_{\preceq}$ and, consequently, either $(x, y) \in P_{\preceq}$, or $(y, x) \in P_{\preceq}$. If $(x, y) \in P_{\preceq}$, it follows from $(y, x) \in S$, $(x, y) \in P_{\preceq}$ and $(y, y) \in S$ that $(y, y) \in S \circ P_{\preceq} \circ S$, but it is impossible since $S \circ P_{\preceq} \circ S$ is irreflexive. Using the similar argument, we conclude that the case $(y, x) \in P_{\preceq}$ is also impossible. It proves that S is a subset of I_{\preceq} . \square

Recall that a binary relation $G \subset X \times X$ is said to be *acyclic* if for any finite collection of elements $x_1, x_2, \dots, x_n \in X$ it follows from $(x_i, x_{i+1}) \in G$, $i = 1, \dots, n-1$, that $(x_n, x_1) \notin G$. The *transitive hull* of a binary relation G is the smallest transitive relation $TH(G)$ containing G . There holds the equality $TH(G) = \cup\{G^n \mid n \in \mathbb{N}\}$, where \mathbb{N} stands for the set of natural numbers and $G^n := \underbrace{G \circ G \circ \dots \circ G}_n$. It immediately follows from the latter equality that a binary relation G is acyclic if and only if its transitive hull $TH(G)$ is irreflexive (or, equivalently, if and only if $TH(G)$ is asymmetric).

Theorem 5 *Let S be an equivalency relation on a set X and \preceq a partial order defined on the same set X . Then the following statements are equivalent:*

- (i) *there exists a total preorder S -extension of \preceq ;*
- (ii) *the composition $S \circ P_{\preceq} \circ S$ is acyclic.*

Proof (i) \implies (ii) Let a total preorder \preceq be a total preorder S -extension of a partial order \preceq . Assume that the composition $S \circ P_{\preceq} \circ S$ is not acyclic and let the collection $x_1, x_2, \dots, x_m \in X$, $m \geq 2$, hold $(x_i, x_{i+1}) \in S \circ P_{\preceq} \circ S$, $i = 1, \dots, m-1$, and $(x_m, x_1) \in S \circ P_{\preceq} \circ S$. Then there exist collections $y_1, y_2, \dots, y_m \in X$ and $z_1, z_2, \dots, z_m \in X$ such that $(x_i, y_i) \in S$, $(y_i, z_i) \in P_{\preceq}$, $(z_i, x_{i+1}) \in S$, $i = 1, 2, \dots, m-1$, and $(x_m, y_m) \in S$, $(y_m, z_m) \in P_{\preceq}$, $(z_m, x_1) \in S$. The inclusion $P_{\preceq} \subset P_{\preceq}$ implies that $(y_i, z_i) \in P_{\preceq}$, $i = 1, \dots, m$. Since $S = R_{\preceq}$, we conclude from Proposition 1 that $P_{\preceq} \circ S = S \circ P_{\preceq} = P_{\preceq}$. Hence, it follows from $(x_i, y_i) \in S$, $(y_i, z_i) \in P_{\preceq}$, $(z_i, x_{i+1}) \in S$, $i = 1, 2, \dots, m-1$, that $(x_i, x_{i+1}) \in P_{\preceq}$, $i = 1, 2, \dots, m-1$, whence, due to the transitivity of P_{\preceq} , we obtain $(x_1, x_m) \in P_{\preceq}$. On the other hand, from $(x_m, y_m) \in S$, $(y_m, z_m) \in P_{\preceq}$, $(z_m, x_1) \in S$, using the equalities $P_{\preceq} \circ S = S \circ P_{\preceq} = P_{\preceq}$, we deduce $(x_m, x_1) \in P_{\preceq}$. This is a contradiction because P_{\preceq} is asymmetric. It proves that $S \circ P_{\preceq} \circ S$ should be acyclic.

(ii) \implies (i) The relation $S \circ P_{\preceq} \circ S$ is acyclic if and only if its transitive hull $TH(S \circ P_{\preceq} \circ S)$ is asymmetric. Using the equality $TH(S \circ P_{\preceq} \circ S) = \cup\{(S \circ P_{\preceq} \circ S)^n \mid n \in \mathbb{N}\}$ it is not difficult to verify that $S \circ TH(S \circ P_{\preceq} \circ S) = TH(S \circ P_{\preceq} \circ S) \circ S = TH(S \circ P_{\preceq} \circ S)$. Hence, due to Theorem 3 there exists a total preorder \preceq on X which is a total preorder S -extension of a partial order $E \cup TH(S \circ P_{\preceq} \circ S)$ (recall that $E := \{(x, x) \in X \times X \mid x \in X\}$ is the equality relation on X). Since $P_{\preceq} \subset TH(S \circ P_{\preceq} \circ S)$, the preorder \preceq is also a total preorder S -extension of a partial order \preceq . \square

Theorem 6 *Let \preceq be a partial order on a set X and S an equivalency relation defined on the same set X . If the composition $S \circ P_{\preceq} \circ S$ is acyclic then the intersection of all total preorder S -extension of the partial order \preceq is the preorder $S \cup TH(S \circ P_{\preceq} \circ S)$, that is the preorder whose asymmetric part is the transitive hull of $S \circ P_{\preceq} \circ S$ and whose symmetric part coincides with S .*

Proof Since the asymmetric part of every total preorder S -extension of the partial order \preceq is transitive and contains $S \circ P_{\preceq} \circ S$, it also contains the transitive hull of $S \circ P_{\preceq} \circ S$. Hence, every total preorder S -extension of the partial order \preceq is at the same time a total preorder S -extension of the partial order $E \cup TH(S \circ P_{\preceq} \circ S)$. Conversely, it follows from $P_{\preceq} \subset S \circ P_{\preceq} \circ S$ that every total preorder S -extension of the partial order $E \cup TH(S \circ P_{\preceq} \circ S)$ is a total preorder S -extension of the partial order \preceq . Since $S \circ TH(S \circ P_{\preceq} \circ S) = TH(S \circ P_{\preceq} \circ S) \circ S = TH(S \circ P_{\preceq} \circ S)$, we conclude from the second statement of Theorem 3 that the intersection of all total preorder S -extension of the partial order \preceq coincides with the preorder $S \cup TH(S \circ P_{\preceq} \circ S)$. \square

Given a partial order \preceq on X , by the symbol $\Sigma(\preceq)$ (respectively, $\Sigma^*(\preceq)$) we denote the collection consisting of all equivalency relations S defined on X such that the composition $S \circ P_{\preceq} \circ S$ is irreflexive (respectively, $S \circ P_{\preceq} \circ S$ is acyclic). Clearly, $\Sigma^*(\preceq)$ is a subcollection of the collection $\Sigma(\preceq)$. It also follows from Proposition 2 and Theorem 5 that $S \in \Sigma(\preceq)$ if and only if $S \subset I_{\preceq}$ and $S \in \Sigma^*(\preceq)$ is equivalent to the existence of a total preorder S -extension of the partial order \preceq .

Theorem 7 *Let S be an equivalency relation defined on a set X . A partial order \preceq defined on the same set X has a unique total preorder S -extension if and only if S is maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$ and the transitive hull of the composition $S \circ P_{\preceq} \circ S$ is negatively transitive.*

Proof Let \preceq be a unique total preorder S -extension of the partial order \preceq . Suppose to the contrary that S is not maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$. Then there exists an equivalency relation S' in $\Sigma^*(\preceq)$ such that $S \subset S'$, $S \neq S'$. Let \preceq' be an arbitrary total preorder S' -extension of the partial order \preceq . Suppose that $P_{\preceq'} \not\subset P_{\preceq}$. Denote by \preceq^* an arbitrary total preorder S -extension of the partial order $P_{\preceq'} \cup E$. The existence of \preceq^* follows from the inclusion $S \subset S' = R_{\preceq'}$ and Corollary 6. Since $P_{\preceq} \subset P_{\preceq'} \subset P_{\preceq^*}$, then \preceq^* is also a total preorder S -extension of the initial partial order \preceq . It follows from the assumption $P_{\preceq'} \not\subset P_{\preceq}$ and the inclusion $P_{\preceq'} \subset P_{\preceq^*}$ that $P_{\preceq^*} \not\subset P_{\preceq}$. Hence $\preceq^* \neq \preceq$. Since it contradicts the uniqueness of a total preorder S -extension of the partial order \preceq , then the inclusion $P_{\preceq'} \not\subset P_{\preceq}$ is impossible and, consequently, we have $P_{\preceq'} \subset P_{\preceq}$. In this case we define on X the relation $\preceq^\circ := P_{\preceq'} \cup (P_{\preceq}^{-1} \cap S') \cup S$. It is not difficult to verify that \preceq° is a total preorder, which differs from \preceq only on the equivalency classes of S' , where it coincides with the converse relation of \preceq . Since $P_{\preceq} \subset P_{\preceq'} \subset P_{\preceq^\circ} := P_{\preceq'} \cup (P_{\preceq}^{-1} \cap S')$ and $S_{\preceq^\circ} = S$, then \preceq° is a total preorder S -extension of the partial order \preceq which differs from \preceq . Again we get the contradiction to the uniqueness of a total preorder S -extension of the partial order \preceq . This completes the proof that S is maximal (in inclusion) in the subcollection $\Sigma^*(\preceq)$.

It remains to prove that $TH(S \circ P_{\preceq} \circ S)$ is negatively transitive. Since \preceq is the unique total preorder S -extension of the partial order \preceq then due to Theorem 6 we conclude that $P_{\preceq} = TH(S \circ P_{\preceq} \circ S)$. Notice now that P_{\preceq} is the asymmetric part of the total preorder \preceq and therefore it is negatively transitive. Hence, $TH(S \circ P_{\preceq} \circ S)$ is negatively transitive too.

For the converse, notice that the assumption $S \in \Sigma^*(\preceq)$ is equivalent to the asymmetry property of the transitive hull $TH(S \circ P_{\preceq} \circ S)$. It implies that the relation $\preceq := S \cup TH(S \circ P_{\preceq} \circ S)$ is the preorder. Since $TH(S \circ P_{\preceq} \circ S)$ is negatively transitive, the indifference relation I_{\preceq} corresponding to \preceq is transitive and hence I_{\preceq} is an equivalency relation. Then the relation $I_{\preceq} \cup TH(S \circ P_{\preceq} \circ S)$ is a total preorder and, moreover, it follows from $P_{\preceq} \subset TH(S \circ P_{\preceq} \circ S)$ that $I_{\preceq} \in \Sigma^*(\preceq)$. Thus, the preorder $\preceq := S \cup TH(S \circ P_{\preceq} \circ S)$ is total and consequently it is a total preorder S -extension of the partial order \preceq . From Theorem 6 we conclude that there are no other total preorder S -extensions of the partial order \preceq . \square

Theorem 8 *Let \preceq be a partial order on X and S an equivalency relation defined on the same set X . The relation $S \cup (S \circ P_{\preceq} \circ S)$ is the unique total preorder S -extension of the partial order \preceq if and only if S belongs to the subcollection $\Sigma^*(\preceq)$ and is maximal (by inclusion) in the collection $\Sigma(\preceq)$.*

Proof Assume that an equivalence relation S belongs to $\Sigma^*(\preceq)$ and is maximal (by inclusion) in $\Sigma(\preceq)$. First we prove that for any $x, y \in X$ there holds exactly one alternative of the following three ones:

$$(x, y) \in S, (x, y) \in S \circ P_{\preceq} \circ S, (y, x) \in S \circ P_{\preceq} \circ S.$$

Choose $x, y \in X$ with $(x, y) \notin S$. If $(x, y) \in P_{\preceq}$ or $(y, x) \in P_{\preceq}$, then $(x, y) \in S \circ P_{\preceq} \circ S$ or $(y, x) \in S \circ P_{\preceq} \circ S$, respectively. Let $(x, y) \in I_{\preceq} \setminus S$ and let $[x]_S$ and $[y]_S$ be the equivalency classes of S containing x and y , respectively. Since S is maximal (in inclusion) in $\Sigma(\preceq)$, there exist $x_1 \in [x]_S$ and $y_1 \in [y]_S$ such that either $(x_1, y_1) \in P_{\preceq}$, or $(y_1, x_1) \in P_{\preceq}$. Indeed, if $(x_1, y_1) \notin P_{\preceq}$ and $(y_1, x_1) \notin P_{\preceq}$ for any $x_1 \in [x]_S$ and $y_1 \in [y]_S$, then the equivalency relation S' defined on X by

$$(u, v) \in S' \iff \text{either } (u, v) \in S \text{ or } u, v \in [x]_S \cup [y]_S.$$

satisfies $S' \subset I_{\preceq}$. Since $S \subset S'$, $S \neq S'$, it contradicts maximality (by inclusion) of S in $\Sigma(\preceq)$.

Thus, for any pair $(x, y) \in I_{\preceq} \setminus S$ there exist $x_1 \in [x]_S$ and $y_1 \in [y]_S$ such that either $(x_1, y_1) \in P_{\preceq}$, or $(y_1, x_1) \in P_{\preceq}$.

If $(x_1, y_1) \in P_{\preceq}$ is the case then it follows from $(x, x_1) \in S$, $(x_1, y_1) \in P_{\preceq}$, $(y_1, y) \in S$ that $(x, y) \in S \circ P_{\preceq} \circ S$. Similarly, in the case when $(y_1, x_1) \in P_{\preceq}$ we get from $(y, y_1) \in S$, $(y_1, x_1) \in P_{\preceq}$, $(x_1, x) \in S$ that $(y, x) \in S \circ P_{\preceq} \circ S$.

The fact that for any $x, y \in X$ there holds exactly one of the three possible alternatives follows from the assumption $S \in \Sigma^*(\preceq)$ or, equivalently from the acyclicity of $S \circ P_{\preceq} \circ S$.

Assume now that $\preceq \subset X \times X$ is an arbitrary total preorder S -extension of the partial order \preceq (the existence of total preorder S -extensions for \preceq is guaranteed by the assumption that $S \in \Sigma^*(\preceq)$.) It follows from $S_{\preceq} = S$ и $P_{\preceq} \subset P_{\preceq}$ that $S \circ P_{\preceq} \subset S \circ P_{\preceq} = P_{\preceq}$ and then $S \circ P_{\preceq} \circ S \subset P_{\preceq} \circ S = P_{\preceq}$. To prove the converse inclusion let us consider a pair $(x, y) \in P_{\preceq}$. Then $(x, y) \notin S$ and by the assertion proved above we have either $(x, y) \in S \circ P_{\preceq} \circ S$, or $(y, x) \in S \circ P_{\preceq} \circ S$. The latter is impossible because it contradicts the asymmetry property of P_{\preceq} . Hence, $(x, y) \in S \circ P_{\preceq} \circ S$ and we get $P_{\preceq} = S \circ P_{\preceq} \circ S$. Since \preceq is an arbitrary total preorder S -extension of \preceq we conclude that $S \cup (S \circ P_{\preceq} \circ S)$ is the unique total preorder S -extension of the partial preorder \preceq .

To verify the converse, assume that $S \cup (S \circ P_{\preceq} \circ S)$ is a total preorder S -extension of a partial order \preceq . Obviously, $S \in \Sigma^*(\preceq)$. Suppose that $S \subset S'$ for some $S' \in \Sigma(\preceq)$. Then $S \circ P_{\preceq} \circ S \subset S' \circ P_{\preceq} \circ S'$ and $S' \cap (S' \circ P_{\preceq} \circ S') = \emptyset$. Since the preorder $S \cup (S \circ P_{\preceq} \circ S)$ is total, we get that $S' \subset S$. Hence, $S = S'$ and it proves that S is maximal in $\Sigma(\preceq)$. \square

Remark Let X be a real vector space and let a partial order \preceq and an equivalency relation S defined on X be compatible with algebraic operations on X . Then the condition $S \cap P_{\preceq} = \emptyset$ is both necessary and sufficient for the existence of a compatible total preorder S -extension of \preceq . This criterion follows from the Kakutani–Tukey theorem on separation of convex sets by halfspaces (see, for instance, [12, Theorem 1.9.1, p. 12]) and from the duality between compatible total preorders and conic halfspaces [15–16].

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